JOURNAL OF ALGEBRA 24, 25–55 (1973)

Locally Compact Modules*

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Communicated by D. A. Buchsbaum

Received February 4, 1971

1. INTRODUCTION

Pontryagin first worked out the structure and duality theory for locally compact abelian groups in the 1930's. This theory has since played an important role in the modern adelic approach to number theory. Of particular importance to number theory are certain locally compact abelian groups with lattice.

The purpose of this paper is twofold. The first is to generalize Pontryagin's theory to a theory of locally compact modules over appropriate rings. The second is to characterize in general those locally compact modules which contain a lattice and to characterize in particular the adele rings.

The following notations are used throughout this paper.

$\mathscr{R}$ is an arbitrary eommutative ring unless explicitly stated otherwise. $\mathscr{R}$ is always assumed to have the discrete topology.

$\mathscr{Z}$ is either the ring $\mathbf{Z}$ of rational integers or the ring $F[t]$ of polynomials in the indeterminate t, over the finite constant field F. Q is the field of fractions of $\mathscr{Z}$. k is a finite separable algebraic extension of Q. Finally, $\mathcal{O}$ is the integral closure of $\mathscr{Z}$ in k. A field, k, formed in this way is called a global field, and $\mathcal{O}$ is called a ring of integers of k. (We note that $\mathscr{Z}$, Q, and $\mathcal{O}$ are uniquely determined by k in the case of characteristic O but not in the case of characteristic $p > 0$.)

It is well known that a ring of integers of a global field is a Dedekind domain. We will frequently make use of this fact.

We generalize the usual definition of "$\mathscr{R}$-module" to

DEFINITION. We say that M is a *locally compact module over* $\mathscr{R}$ or more simply that M is an *$\mathscr{R}$-module* to mean that: (i) M is a locally compact abelian

* This work was partially supported by the National Science Foundation. Theorem 5 for the special case of $\mathbf{Z}$-modules was included in the author's doctoral dissertation submitted to the Johns Hopkins University in June, 1969.

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group; (ii) M is a module over $\mathscr{R}$ in the usual sense (and unitary if $\mathscr{R}$ has an identity); and (iii) scalar multiplication is continuous.

Two observations: In the case that M is discrete, our definition of "$\mathscr{R}$-module" reduces to the usual one; secondly, any locally compact abelian group is a $\mathscr{Z}$-module in the obvious way.

A more explicit description of the contents of this paper is as follows. In Section 2 Pontryagin's duality theory is generalized to the category of $\mathscr{R}$-modules where $\mathscr{R}$ is any commutative ring. Structure-theory results require more restrictions on the ring. In Section 3 the structure theory of locally compact abelian groups is generalized to modules over the ring, $\mathcal{O}$, of integers of a global field. In Section 4 we characterize those $\mathcal{O}$-modules which contain a lattice (i.e., a discrete submodule with compact quotient). In Section 3 we use the theory of $\mathcal{O}$-modules to obtain several characterizations of the adele rings. Some of these characterizations are entirely new and some are improvements of known results. Section 6 is devoted to an application and some examples. In Section 7, we investigate the duality theory further, showing that the duality functor of Section 2 is representable and showing that for a Dedekind domain, $\mathscr{R}$, the invertible representable functors in the category of $\mathscr{R}$-modules are in two to one (one covariant and one contravariant) correspondence with the ideal classes of $\mathscr{R}$. Finally, in Section 8, we make some remarks about local linear compactness.[1]

2. Duality Theory

Let M and N be $\mathscr{R}$-modules. We say that a function from M to N is a *homomorphism* (or a map) if it is continuous and $\mathscr{R}$-linear. We write $f : M \rightarrow N$ to mean that f is a homomorphism from M to N. Note that not every homomorphism is an open map.

By the *category of $\mathscr{R}$-modules* we mean the category whose objects are $\mathscr{R}$-modules and whose morphisms are homomorphisms. A sequence, $0 \rightarrow A^i \rightarrow B^f \rightarrow C \rightarrow 0$ in this category is called *exact* to mean that it is exact in the usual sense (i.e., i is injective, f is surjective, and $\operatorname{Im} i = \ker f$) and that i and f are both open homomorphisms. This definition of "exactness" is equivalent to the universal mapping definition applied to the category of $\mathscr{R}$-modules.

Let $\mathbf{T}$ be the multiplicative group of complex numbers of absolute value one. Let M be any $\mathbf{Z}$-module (i.e., a locally compact abelian group). A *character*

[1] The idea of trying to characterize the adele rings by studying the structure of locally compact abelian groups with lattice was suggested to me by my thesis advisor, Professor J. I. Igusa. It was my own idea to introduce the notion of locally compact modules in order to carry this out.

of M is a homomorphism from M to $\mathbf{T}$. Let M^* denote the set of all characters of M. For any χ and χ' in M^* let $\chi + \chi'$ be the character of M defined by $(\chi + \chi')(x) = \chi(x) + \chi'(x)$ for all x in M. Put the compact open topology on M^*. It turns out that with this definition of addition and this topology, M^* is itself a $\mathbf{Z}$-module. M^* is called the character group of M.

If $f : M \to N$, then one defines $f^* : M^* \to N^*$ by $f^*(\chi) = \chi \circ f$ for every χ in M^*; it turns out that f^* is a homomorphism as the notation indicates. Moreover, if $f : M \to N$ and $g : N \to A$, then $f^* \circ g^* = (g \circ f)^*$. So $*$ is a contravariant functor from the category of $\mathbf{Z}$-modules to itself.

DEFINITION. Let M be an $\mathscr{R}$-module. Then in particular M is a local compact abelian group and is hence a $\mathbf{Z}$-module, and as such it has a dual group, M^*. We wish to make M^* into an $\mathscr{R}$-module. Each λ in $\mathscr{R}$ may be considered as an endomorphism of M (the endomorphism which sends x to λx) and as such it has a dual, λ^*, which is an endomorphism of M^*. Let $\mathscr{R}$ operate on M^* by defining $\lambda \cdot \chi = \lambda^*(\chi)$ for each λ in $\mathscr{R}$ and χ in M^*. It is easily checked, using the commutativity of $\mathscr{R}$, that this makes M^* into an $\mathscr{R}$-module which we call the *dual module* of M. If M is an $\mathscr{R}$-module, M^* will henceforth denote its dual module.

THEOREM 1. Generalized Pontryagin Duality. *Let $\mathscr{R}$ be a commutative ring. Then*

(i) *$*$ is a contravariant functor from the category of $\mathscr{R}$-modules to itself.*

(ii) *Each $\mathscr{R}$-module, M, can be identified with its second dual, M^{**}, in such a way that any map, f, of $\mathscr{R}$-modules, gets identified with its second dual.*

(iii) *$*$ is an exact functor. More precisely, if $0 \to A \xrightarrow{i} B \xrightarrow{f} C \to 0$ is exact in the category of $\mathscr{R}$-modules, then $0 \to C^* \xrightarrow{f^*} B^* \xrightarrow{i^*} A^* \to 0$ is also exact in the category of $\mathscr{R}$-modules.*

(iv) *M is a compact $\mathscr{R}$-module if and only if M^* is a discrete $\mathscr{R}$-module.*

Proof. For the case of $\mathbf{Z}$-modules this is the well-known Pontryagin duality theorem. A very readable proof may be found in [13]. A proof by means of the theory of Banach Algebra may be found in [12]. The case of arbitrary commutative $\mathscr{R}$ follows easily from the $\mathbf{Z}$-module case and definitions. We remark that the identification of M to M^{**} is defined as follows: for each x in M let x' be the character of M^* defined by $x'(\chi) = \chi(x)$ for each χ in M^*; then $x \to x'$ is the canonical identification of M to M^{**} referred to in (ii) above.

We henceforth assume that M has been identified with M^{**}.

We now give a few corollaries of this theorem, corollaries which will be needed later in this paper.

Suppose that M is an $\mathscr{R}$-module. We say that a subset, N, of M is a *submodule* to mean that N is a subgroup closed under scalar multiplication. We say that N is a *closed submodule* of M to mean than N is a submodule of M and that N is a topologically closed subset of M. It is easily shown that a submodule, N, of M, is itself an $\mathscr{R}$-module (i.e., is locally compact) if and only if N is a closed submodule of M.

DEFINITION. Suppose that M is an $\mathscr{R}$-module and that N is a closed submodule of M. A character, χ, of M is said to annihilate N if $\chi(x) = 0$ for all x in N. The set of all characters of M which annihilate N is called the *annihilator of N with respect to M* or if there is no chance of ambiguity, it is called more simply the *annihilator of N* and is denoted $N_\perp$.

COROLLARY 1. *Suppose that M is an $\mathscr{R}$-module. Then for each closed submodule, N, of M, $N_\perp$ is a closed submodule of M^*. Moreover, $(N_\perp)_\perp = N$ (where M is identified with M^{**}). Thus, $N \leftrightarrow N_\perp$ gives a one to one correspondence between the closed submodules of M and the closed submodules of M^*. Finally, $N_\perp$ is isomorphic to $(M/N)^*$.*

Proof. The corollary follows from Theorem 1(iii) and the fact that $N_\perp$ is the kernel of $i^* : M^* \to N^*$, where i is the canonical injection of N into M.

COROLLARY 2. *Suppose that M is an $\mathscr{R}$-module, and that $M \supset B \supset A$, where A and B are closed submodules. Let $A_\perp$ and $B_\perp$ be the annihilators with respect to M of A and B, respectively. Then $M^* \supset A_\perp \supset B_\perp$. Moreover, $A_\perp/B_\perp$ is isomorphic to $(A/B)^*$.*

Proof. $A_\perp \supset B_\perp$, since a character which annihilates A annihilates B as well.

To prove the second statement, consider the sequence $A \xrightarrow{i} B \xrightarrow{j} M$ where i and j are the canonical inclusions. Considering the dual sequence, $M^* \xrightarrow{j^*} B^* \xrightarrow{i^*} A^*$, one sees that the kernel of i^* is isomorphic to $A_\perp/B_\perp$, since the kernel of $i^* \circ j^* = (i \circ j)^*$ is $A_\perp$ and the kernel of j^* is $B_\perp$. On the other hand, the kernel of i^* is the annihilator of A with respect to B, and hence isomorphic to $(A/B)^*$ by Corollary 1. This proves Corollary 2.

COROLLARY 3. *Suppose that M is an $\mathscr{R}$-module. Then a closed submodule, K of M is compact if and only if its annihilator, $K_\perp$, is open in M^*.*

Proof. Observe that $K_\perp$ is open in M^* if and only if $M^*/K_\perp$ is discrete. Corollary 3 then follows by the last statement of Corollary 1 and Theorem 1(iv).

DEFINITION. Let M be an $\mathscr{R}$-module. M is called *torsion free* if multiplication by λ is an injective endomorphism of M for each λ in $\mathscr{R}$. M is called *divisible* if multiplication by λ is a surjective endomorphism of M for each λ in $\mathscr{R}$.

COROLLARY 4. *Let M be an $\mathscr{R}$-module. Then M is compact and divisible if and only if M^* is torsion free and discrete.*

Proof. If M is any $\mathscr{R}$-module and λ is any element of $\mathscr{R}$, then the annihilator of $(\lambda M)^c$ in M^* is easily seen to be $\{\chi \in M^* : \lambda\chi = 0\}$ (where superscript c denotes topological closure). Suppose that M is compact or equivalently that M^* is discrete. Then $\lambda M = (\lambda M)^c$. Hence, multiplication by λ on M is surjective if and only if multiplication by λ on M^* is injective. The corollary follows.

There is one additional fact about duality which we will frequently use without further reference. $(A \oplus B)^*$ is isomorphic to $A^* \oplus B^*$. This holds because direct sums are preserved by any additive functor.

3. STRUCTURE THEORY

The fundamental structure theorem of locally compact abelian groups (i.e., $\mathbf{Z}$-modules) asserts that any such group is the direct sum of a real vector group and a group which contains a compact open subgroup. This theorem is due essentially to Pontryagin.

In the previous section we generalized the Pontryagin duality theory to modules over any commutative ring. However, generalizing the structure theory requires further restrictions on the ring.

In this section we think of $\mathcal{O}$ as a fixed ring of integers of a global field, k, with $\mathscr{X}$ and Q as in Section 1. We generalize to $\mathcal{O}$-modules the fundamental structure theorem of locally compact abelian groups. Also, we generalize to $\mathcal{O}$-modules some of the more detailed structure theory of locally compact abelian groups.

Important to our theory is the following distinction between the two different types of valuations of k.

DEFINITION. A valuation, v, of k is called *finite with respect to $\mathcal{O}$* if and only if $v(\lambda) \leqslant 1$ for every λ in $\mathcal{O}$. A valuation, v, of k is called *infinite with respect to $\mathcal{O}$* if it is not finite with respect to $\mathcal{O}$. If there is no chance of confusion, we will simply say that v is *finite* or *infinite* as the case may be.

We remark that the term "finite" valuation has been used with different meaning by other authors.

From the well-known classification of the valuations of a global field, it follows that there is precisely one valuation of Q which is infinite with respect to $\mathscr{Z}$, namely the absolute-value valuation in the case of characteristic O and the "degree" valuation in the case of characteristic $p > 0$. Moreover, a valuation of k is infinite with respect to $\mathcal{O}$ if and only if its restriction to $\underset{\sim}{Q}$ is infinite with respect to $\mathscr{Z}$. Consequently, we see that there are only finitely many valuations of k which are infinite with respect to $\mathcal{O}$ (and that in the case of characteristic O they are precisely the archimedean valuations of k).

By a *completion of* k we mean the completion of k with respect to some valuation of k. We call a completion of k finite or infinite (with respect to $\mathcal{O}$) according as the corresponding valuation is finite or infinite (with respect to $\mathcal{O}$). Note that any completion of k is a locally compact field which contains $\mathcal{O}$ as a subring and is hence an $\mathcal{O}$-module. These $\mathcal{O}$-modules play an important role in much of what is to follow.

THEOREM 2. Fundamental Structure Theorem of $\mathcal{O}$-Modules. *Let $\mathcal{O}$ be a ring of integers of a global field k. Then any $\mathcal{O}$-module, M, can be written as*

$$M = M_2 \oplus R,$$

where M_2 contains a compact open $\mathcal{O}$-submodule and where R is the direct sum of finitely many infinite (with respect to $\mathcal{O}$) completions of k.

This theorem shows among other things that the reals play such a special role in fundamental structure theorem of locally compact abelian groups (i.e., $\mathbf{Z}$-modules) precisely because they are the unique archimedean completion of the field of fractions of $\mathbf{Z}$.

Before proving the fundamental structure theorem, we give a couple of preliminary lemmas.

DEFINITION. Let $\underset{\sim}{Q}_\infty$ be the unique infinite (with respect to $\mathscr{Z}$) completion of $\underset{\sim}{Q}$. Let $k_\infty = \underset{\sim}{Q}_\infty \otimes_{\underset{\sim}{Q}} k$.

It is well known [17] and easily proved that k_∞ is isomorphic as a ring and hence as an $\mathcal{O}$-module to the direct sum of the infinite (with respect to $\mathcal{O}$) completions of k, each taken once.

The map $\lambda \to 1 \otimes \lambda$ embeds k, and hence $\mathcal{O}$, as subrings of k_∞.

LEMMA 1. (i) *If k_v is an infinite completion of k, then $k_v{}^*$ is isomorphic as an $\mathcal{O}$-module to k_v.*

(ii) *$\mathcal{O}^*$ is isomorphic to $k_\infty/\mathcal{O}$.*

Proof. (i) is well known [17, p. 41]. (ii) can be proved by reducing to the case $k = \underset{\sim}{Q}$.

LEMMA 2. *If N is an open divisible submodule of the $\mathcal{O}$-module, M, then M decomposes as an $\mathcal{O}$-module into the direct sum of N and a discrete module.*

Proof. For abstract (i.e., forgetting topology) modules over Dedekind domains, it is well known [7] that a divisible module is a direct summand of any containing module. From the openness of N it follows that its complementary summand is discrete and that the decomposition is topological. Lemma 2 is proved.

Now we turn to the proof of the fundamental structure theorem. In order to prove this theorem, it is sufficient to prove it for the case of compactly generated $\mathcal{O}$-modules. Indeed, let M be any $\mathcal{O}$-module. Then M contains a compactly generated submodule, say N, which is open in M; for instance, the submodule generated by any compact neighborhood of O in M. Assume for the moment that the theorem holds for N. Then $N = N_2 \oplus R$ where N_2 contains a compact open submodule, say K, and R is the direct sum of infinite completions of k. Then $K \oplus R$ is an open submodule of M. Hence, R is an open divisible submodule of M/K. By Lemma 2 $M/K = L \oplus R$ for some discrete submodule, L, of M/K. Let M_2 be the inverse image of L by the canonical map, $M \to M/K$. Then $M = M_2 \oplus R$ and M_2 contains K as a compact open submodule. Hence, the structure theorem holds for M if it holds for N. Thus, it is sufficient to prove the fundamental structure theorem for the case of compactly generated $\mathcal{O}$-modules. For this case, we actually have the stronger result

PROPOSITION 1. *Let $\mathcal{O}$ be the ring of integers of a global field. Then any compactly generated $\mathcal{O}$-module, M, can be written as*

$$M = K \oplus D \oplus R,$$

where K is compact, D is discrete and is the direct sum of finitely many ideals of $\mathcal{O}$, and R is the direct sum of finitely many infinite completions of k.

For the special case $\mathcal{O} = \mathbf{Z}$, this is the well-known theorem of Pontryagin which appears as Theorem 51 in [13]. In proving Theorem 51, Pontryagin has available to him the duality theory only for compact and discrete $\mathbf{Z}$-modules, since he uses Theorem 51 to establish the duality theory for arbitrary $\mathbf{Z}$-modules. However, we are able to regard the full duality theory for $\mathcal{O}$-modules as already established by one of the proofs for the $\mathbf{Z}$-module case combined with Theorem 1. In the proof that follows we draw heavily from Pontryagin's proof of Theorem 51, but we make a considerable simplification by use of the full duality theorem. On the other hand we have to make a few additional considerations in order to deal with $\mathcal{O}$-modules instead of just $\mathbf{Z}$-modules.

Proof of Proposition 1. (1) If a is any element of a $\mathcal{Z}$-module, M, then $\mathcal{Z} \cdot a$ is either discrete in M or else $\mathcal{Z} \cdot a$ has compact closure in M.

For the case $\mathscr{X} = \mathbf{Z}$ this appears as Lemma 1 in [13]. For the case $\mathscr{X} = F[t]$, the proof is as follows. There is no loss of generality in replacing M by the closure of $\mathscr{X} \cdot a$ in M. Hence, we may and do assume that $\mathscr{X} \cdot a$ is dense in M. Let V be a symmetric open neighborhood of zero in M which generates M and which has compact closure in M. Then $M = F[t] \cdot a + V$ since $F[t] \cdot a$ is dense in M. Then, by the compactness of $t \cdot V + V + V$ one has

$$(tV + V + V) \subset V + F \cdot a + Fta + \cdots + Ft^n a \qquad (\#)$$

for some finite n. Suppose that $F[t] \cdot a$ is not discrete. Then $V \cap F[t] \cdot a$ must be infinite and hence contain $g(t) \cdot a$ for some polynomial, $g(t)$, of degree, say s, which is larger than n. Then

$$t^s a = \lambda_0 a + \lambda_1 ta + \cdots + \lambda_{s-1} t^{s-1} a + v,$$

where all λ_i are in F and where v is in V. It follows from this and $(\#)$ that $N = V + Fat \cdots + Ft^s a$ is left invariant by the operation of t and that $N + N \subset N$. Hence N is a submodule of M. Moreover, N equals M, since it contains both V and a, and N is compact since it is the union of finitely many translates of V. Therefore, M is compact. This proves (1).

(2) If M is any compactly generated $\mathscr{X}$-module, then M contains a finitely generated discrete Z-module, say L, such that M/L is compact.

To prove this statement let V be an open neighborhood of O in M, which generates M and whose closure in M, say V^c, is compact. By the compactness of $V^c + V^c + tV^c$, there is a finite set, $S = \{a_1, \ldots, a_n\}$, of elements of M such that $\bigcup_i (a_i + V)$ covers $V^c + V^c + tV^c$. Let A be the $\mathscr{X}$-module of M generated by S. Then clearly $A + V = M$. Let B be a subset of S maximal with respect to the property that the $\mathscr{X}$-submodule of M which B generates is discrete in M. Let L be the $\mathscr{X}$-submodule of M generated by B. L is discrete and finitely generated. If we can show that M/L is compact, then the proof of (2) will be complete. Let f be the canonical map, $f: M \to M/L$. It is easily seen by (1) and the maximality of B that for each a_i in S $\mathscr{X} \cdot f(a_i)$ has compact closure in M/L. It follows that $f(A)$ has compact closure in M/L. Hence, $M/L = f(A) + f(V)$ is compact. This proves (2).

(3) If M is a compactly generated $\mathscr{X}$-module then M^* contains an open $\mathscr{X}$-submodule, say N, such that N is isomorphic to $Q_\infty{}^r \oplus (Q_\infty/\mathscr{X})^s$ for some finite r and s.

Indeed, let L be a finitely generated discrete $\mathscr{X}$-submodule of M such that M/L is compact as guaranteed by (2). Then L is the direct sum of a finite $\mathscr{X}$-module and finitely many copies of $\mathscr{X}$. By making L smaller if necessary, we may and do assume that L is isomorphic to $\mathscr{X}^m$ for some finite m. Then $L_\perp$ is a discrete $\mathscr{X}$-submodule of M^*; it is discrete since it is the dual of the compact module, M/L. Moreover, $M/L_\perp$ is isomorphic $(Q_\infty/\mathscr{Z})^m$ since it is the dual of L and by Lemma 1. Let f be the canonical projection $f: M^* \to M^*/(L_\perp)$. Let g be a projection from $Q_\infty{}^m$ onto $M^*/L_\perp$ such that the kernel of g is $\mathscr{Z}^m$. f and g are local isomorphisms since they both have discrete kernels. Hence $f^{-1} \circ g$ defines a local isomorphism of some neighborhood, say V, of O in $Q_\infty{}^m$ onto some neighborhood of O in M. Extend $f^{-1} \circ g$ to all of $Q_\infty{}^m$ as follows: for each y in $Q_\infty{}^m$, $y = \lambda x$ for some λ in $\mathscr{Z}$ and some x in V; define $f^{-1} \circ g(y)$ to be $\lambda \cdot f^{-1} \circ g(x)$. It is easily checked that $f^{-1} \circ g$ is then a well-defined map from $Q_\infty{}^m$ onto an open submodule, say N, of M^*. Moreover, N is isomorphic to $Q_\infty{}^r + (Q_\infty/\mathscr{Z})^s$, where $r + s = m$, since the kernel of g, being contained in $\mathscr{Z}^m$, is isomorphic to $\mathscr{Z}^s$ for some $s \leqslant m$. This proves (3).

(4) If R is an $\mathcal{O}$-module which is isomorphic as a $\mathscr{Z}$-module to $Q_\infty{}^r$, then R is the direct sum of finitely many infinite completions of k.

Indeed, let λ be an element of $\mathcal{O}$ which is a primitive element of k over Q [i.e., $k = Q(\lambda)$]. Q_∞ is the unique infinite completion of Q and R considered as a $\mathscr{Z}$-module is just $Q_\infty{}^r$. λ, considered as an operator on the $\mathscr{Z}$-module, $Q_\infty{}^r$, is $\mathscr{Z}$-linear since $\mathcal{O}$ is a commutative ring. It is easily seen from this that λ is Q-linear as well. Then from the continuity of λ and the fact that Q is dense in Q_∞ it follows that λ is a Q_∞-linear operator on $Q_\infty{}^r$. Thus, if R is regarded as a vector space over Q_∞, then λ is linear transformation of R. Then λ has a minimal polynomial, say $m(x)$, in the sense of linear algebra [i.e., $m(x)$ is the minimal polynomial in $Q_\infty[x]$ for which $m(\lambda)$ operates trivially on R]. Moreover, λ, considered as an element of k, has a minimal polynomial, say $p(x)$, over Q [i.e., $p(x)$ is the minimal polynomial in $Q[x]$ for which $p(\lambda) = 0$ in k]. Then clearly $m(x)$ divides $p(x)$. But $p(x)$ factors into distinct irreducible factors over Q_∞ since k is assumed to be a separable extension of Q. Therefore, $m(x)$ factors over Q_∞ into distinct irreducible factors, say $m(x) = \Pi m_i(x)$. Then, by the rational decomposition theorem of linear algebra $Q_\infty{}^r$ factors into the direct sum of λ-invariant subspaces each of which is isomorphic to $Q_\infty[\lambda]/(m_i(\lambda))$ for some i [where λ is an indeterminate and $(m_i(\lambda))$ is the ideal generated by $m_i(\lambda)$]. But a λ-invarient $\mathscr{Z}$-submodule of R is just an $\mathcal{O}$-submodule of R. Moreover $Q_\infty[\lambda]/(m_i(\lambda))$ is a finite field extension of Q_∞ which contains $Q(\lambda) = k$ as a dense subfield. Since the infinite valuation on Q_∞ is easily extended to a valuation on a finite algebraic extension of Q_∞, it follows that $Q_\infty[\lambda]/(m_i(\lambda))$ is the completion of $k = Q(\lambda)$ with respect to some infinite valuation. Therefore, R is the sum of $\mathcal{O}$-submodules each of which is an infinite completion of k. This proves (4).

(5) Now we are ready to complete the proof of the proposition. Let M be any compactly generated $\mathcal{O}$-module. Then M is compactly generated over $\mathscr{Z}$ as well, since $\mathcal{O}$ has finite rank over $\mathscr{Z}$. Then, by (3), M^* contains an

open $\mathscr{Z}$-submodule, N, which is isomorphic as a $\mathscr{Z}$-module to $Q_\infty{}^r \oplus (Q_\infty/Z)^s$ with r and s finite. Then any λ in $\mathcal{O}$ leaves N invariant since N is the unique minimal open $\mathscr{Z}$-submodule of M^*. Thus, N is an $\mathcal{O}$-submodule of M^*. Moreover, N is a divisible $\mathcal{O}$-module; in fact N is divisible as a $\mathscr{Z}$-module and for any nonzero λ in $\mathcal{O}$, $\lambda\lambda'$ is a nonzero element of $\mathscr{Z}$ for some λ' in $\mathcal{O}$, it follows that N is divisible as an $\mathcal{O}$-module. Therefore, by Lemma 2, M^* is the direct sum of N and a discrete $\mathcal{O}$-submodule. Hence, by Theorem 1(iv), M is the direct sum of N^* and a compact $\mathcal{O}$-submodule, say K. But N^* is $\mathscr{Z}$-isomorphic to $Q_\infty{}^r \oplus \mathscr{Z}^s$ by Lemma 1. $Q_\infty{}^r$ is an $\mathcal{O}$-submodule of N^* and a direct summand of N^* by the same argument that was just applied to N. Thus $M = K \oplus R \oplus D$, where K is a compact $\mathcal{O}$-submodule of M, R is an $\mathcal{O}$-submodule which is $\mathscr{Z}$-isomorphic to $Q_\infty{}^r$, and D is an $\mathcal{O}$-submodule which is $\mathscr{Z}$-isomorphic to $\mathscr{Z}^s$ with r and s finite. But R is the direct sum of infinite completions of k by (4) and D, being a finitely generated torsion-free module over the Dedekind Domain, $\mathcal{O}$, is by the theorem of Steinitz [cf. [7]), the direct sum of ideals in $\mathcal{O}$. The proposition is proved.

Now we go on to discuss some of the more detailed structure theory of $\mathcal{O}$-modules.

Definition. Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}$. Let M be an $\mathcal{O}$-module. For an element, x, in M we say that $\lim_h \mathfrak{p}^h x = 0$ to mean that for any neighborhood, V, of 0 in M, $(\mathfrak{p}^h) \cdot x \subset V$ for all sufficiently large natural numbers, h. We say that M is a *$\mathfrak{p}$-primary $\mathcal{O}$-module* to mean that $\lim_h \mathfrak{p}^h x = 0$ for all x in M.

Definition. We say that an $\mathcal{O}$-module, M, is a *topological torsion* $\mathcal{O}$-module if the union of compact submodules of M is equal to M and the intersection of the open submodules of M is trivial.

In the case that M is discrete this definition of "$\mathfrak{p}$-primary" ("topological torsion") reduces to the usual definition of "$\mathfrak{p}$-primary" ("torsion").

For equivalent definitions of "topological torsion" and "$\mathfrak{p}$-primary" in the special case of Z-modules, see [14].

Definition. Suppose that M_i is an $\mathcal{O}$-module which contains a compact open submodule, K_i, for each i in some index set, I. For x in $\Pi_i M_i$, let x_i denote the M_i coordinate of x. Define the *restricted product of M_i with respect to K_i over all i in I* to be the set of all those x in $\Pi_i M_i$ for which x_i is in K_i for almost all i. Then the restricted product with addition and scalar multiplication defined coordinate-wise is abstractly an $\mathcal{O}$-module which contains $\Pi_i K_i$ as a submodule. Put a topology on the restricted product by taking $\Pi_i K_i$ with the product topology to be an open submodule. Denote this restricted product by $\Pi_i(M_i, K_i)$. Then, clearly $\Pi_i(M_i, K_i)$ is an $\mathcal{O}$-module which contains $\Pi_i K_i$ as a compact open submodule. On occasion we will write $\Pi_i(M_i\,,\ )$ to indicate the restricted product with respect to some unspecified compact open submodule.

Note that if all M_i are discrete and all K_i are taken to be 0, then the restricted product becomes the ordinary direct sum of discrete modules. At the other extreme, if all M_i are compact and each K_i is taken to be M_i, then the restricted product becomes the ordinary product with the product topology.

It can be shown that $(\Pi_i(M_i\,, K_i))^*$ is isomorphic to $\Pi_i(M_i{}^*, (K_i)_\perp)$.

THEOREM 3. Second Structure Theorem of $\mathcal{O}$-modules. *Let M be an $\mathcal{O}$-module. Let $M = M_2 \oplus R$ where M_2 contains a compact open submodule and R is the sum of finitely many infinite completions of k. Let M_1 be the union of all compact submodules of M. Let M_0 be the intersection of all open submodules of M_1. Let $M_3 = M_1/M_0$. Then*

(i) $M_2 \supset M_1 \supset M_0$.

(ii) *M_2/M_1 is a torsion-free discrete $\mathcal{O}$-module. Moreover M_2/M_1 is uniquely determined (up to isomorphism) by M independent of the choice of M_2.*

(iii) *$M_3 = M_1/M_0$ is a topological torsion $\mathcal{O}$-module.*

(iv) *M_0 is compact divisible.*

(v) *For each prime ideal, $\mathfrak{p}$, in $\mathcal{O}$, M_3 contains a unique maximal $\mathfrak{p}$-primary submodule, say $M^{\mathfrak{p}}$.*

$M_3 = \Pi_{\mathfrak{p}}(M^{\mathfrak{p}}, K \cap M^{\mathfrak{p}})$ where the product is taken over all primes $\mathfrak{p}$, and where K is any compact open submodule of M_3.

Proof. If K is any compact submodule of M, then the image of K under the canonical map, $M \to M/M_2 = R$, must be 0 since R contains no nontrivial compact submodules. It follows that M_2 contains every such K, and hence M_2 contains M_1. This establishes (i).

Let K be a compact open submodule of M_2. Then $K \subset M_1$ and it is easily seen that M_1/K is the torsion submodule of the discrete module M_2/K. It follows that M_2/M_1 is torsion free and discrete. Moreover, $M_1 + R$ is a uniquely determined submodule of M, since it equals $M_1 + (M_0 + R)$, where M_1 is the union of all compact submodules of M and $M_0 + R$ is the intersection of all open submodules of M. Therefore M_2/M_1, being isomorphic to $M/(M_1 + R)$, is uniquely determined up to isomorphism by M. This proves (ii).

(iii) follows from the fact the image of a compact (resp. open) submodule under the canonical map, $M_1 \to M_1/M_0$, is compact (resp. open).

(iv) is an immediate consequence of (ii), Corollary 5 of Theorem 1, and part (iii) of the next theorem.

The proof of (v) requires a bit more work. It is well known that any discrete torsion module over a Dedekind Domain decomposes into the direct sum of $\mathfrak{p}$-primary submodules. Elementary considerations show that the dual of a $\mathfrak{p}$-primary module is again $\mathfrak{p}$-primary. It follows by this and Theorem 1(iv) that any compact torsion module decomposes into the product of $\mathfrak{p}$-primary components. In particular for any compact open submodule, K, of M_3, one can write $K = \Pi_{\mathfrak{p}} K^{\mathfrak{p}}$ where each $K^{\mathfrak{p}}$ is $\mathfrak{p}$-primary.

Now put

$$M^{\mathfrak{p}} = \{x \in M_3 \mid \lim_h \mathfrak{p}^h x = 0\}.$$

Then for any compact open submodule, K, of M_3, $M^{\mathfrak{p}} \cap K = K^{\mathfrak{p}}$. Indeed, if x is in $M^{\mathfrak{p}} \cap K$ then for any prime, $\mathfrak{q}$, different from $\mathfrak{p}$, the projection of x onto $K^{\mathfrak{q}}$, call it y, must satisfy both $\lim_h \mathfrak{p}^h y = 0$ and $\lim_h \mathfrak{q}^h y = 0$. It follows from this that y must be in every compact open submodule of M_3 and hence must be 0. This shows that x is in $K^{\mathfrak{p}}$. Inclusion in the other direction is obvious. Thus $M^{\mathfrak{p}} \cap K = K^{\mathfrak{p}}$ as asserted. We have shown that $K = \Pi_{\mathfrak{p}}(M^{\mathfrak{p}} \cap K)$ for any compact open submodule K of M_3.

Let K be any compact open submodule of M_3. Let x be an arbitrary element of M_3. We will show that x can be written uniquely as $x = \Sigma_{\mathfrak{p}} x_{\mathfrak{p}}$ where each $x_{\mathfrak{p}}$ is in $M^{\mathfrak{p}}$ and almost all $x_{\mathfrak{p}}$ are in $M^{\mathfrak{p}} \cap K$. Indeed, let K' be a compact open submodule of M_3 which contains both x and K. Then $K' = \Pi_{\mathfrak{p}}(M^{\mathfrak{p}} \cap K')$ by what we have already shown. Moreover, $M^{\mathfrak{p}} \cap K = M^{\mathfrak{p}} \cap K$ for almost all $\mathfrak{p}$ since K has finite index in K'. It follows that $x = \Sigma_{\mathfrak{p}} x_{\mathfrak{p}}$ where each $x_{\mathfrak{p}}$ is in $M^{\mathfrak{p}}$ and almost all $x_{\mathfrak{p}}$ are in $M^{\mathfrak{p}} \cap K$. For uniqueness, suppose that $\Sigma_{\mathfrak{p}} x_{\mathfrak{p}} = \Sigma_{\mathfrak{p}} y_{\mathfrak{p}}$. Take a compact open submodule, K'', of M_3 which contains both $x_{\mathfrak{p}}$ and $y_{\mathfrak{p}}$ for all $\mathfrak{p}$. Then by the uniqueness of such sums within K'' it follows that $x_{\mathfrak{p}} = y_{\mathfrak{p}}$ for all $\mathfrak{p}$. We have shown that x has the unique expansion as asserted. It follows that $M_3 = \Pi_{\mathfrak{p}}(M^{\mathfrak{p}}, M^{\mathfrak{p}} \cap K)$.

$M^{\mathfrak{p}}$ is the unique maximal $\mathfrak{p}$-primary submodule of M_3; this is clear from its definition. The proof of (v) is complete.

DEFINITION. Let M be any $\mathcal{O}$-module. With notations as in Theorem 3, we make the following definitions. M_2/M_1 is called the *torsion-free discrete part of M*. $M_1/M_0 = M_3$ is called the *topological torsion part of M*. M_0 is called the *compact divisible part of M*. Finally R is called the *infinite part of M*.

We remark that in the special case that M is discrete, the topological torsion part of M becomes the ordinary maximal torsion submodule and Theorem 3(v) becomes the usual decomposition thereof.

For the special case of Z-modules, the decomposition (v) of a topological torsion group into its $\mathfrak{p}$-primary components was known by Braconnier and Dieudonne [1] and also by Vilenkin [16].

THEOREM 4. *Let M be any $\mathcal{O}$-module. Let M_0, M_1, M_2, and R be as in Theorem 3. Then*

(i) $M^* \cong (M_2)^* \oplus R$ *where $M_2{}^*$ contains a compact open submodule and R is the infinite part of M^*.*

(ii) *Consider M_0 and M_1 as submodules of M_2. The annhiilator of M_0 in $(M_2)^*$ is $(M^*)_1$ and the annihilator of M_1 in $(M_2)^*$ is $(M^*)_0$.*

(iii) *The dual of the compact divisible (resp. torsion-free discrete) part of M is the torsion-free discrete (resp. compact divisible) part of M^*.*

(iv) *The dual of the topological torsion part of M is the topological torsion part of M^*.*

(v) *The dual of the $\mathfrak{p}$-primary part of M is the $\mathfrak{p}$-primary part of M^*.*

Proof. (i) $(M_2)^*$ contains a compact open submodule, namely the annihilator of any compact open submodule of M_2, because of Corollary 3 of Theorem 1, also $R^* \cong R$ by Lemma 1. Statement (i) follows.

(ii) follows from Corollary 3 of Theorem 1.

(iii) follows from (ii) and the last statement of Corollary 1 of Theorem 1.

Note that with (iii) established, part (iv) of the previous theorem follows and we are justified in calling M_0 the compact divisible part of M.

(iv) follows from (ii) and Corollary 2 of Theorem 1.

Now we prove (v). Put $M_3 = \Pi_{\mathfrak{p}}(M^{\mathfrak{p}},\ )$ as in Theorem 3(v). Then $(M_3)^* = \Pi_{\mathfrak{p}}(M^{\mathfrak{p}*},\ )$. For each $\mathfrak{p}$, $M^{\mathfrak{p}*}$ is $\mathfrak{p}$-primary; in fact elementary considerations show that the dual of any $\mathfrak{p}$-primary module is again $\mathfrak{p}$-primary. Moreover, if x is in $(M_3)^*$ and $\lim_h \mathfrak{p}^h x = 0$, then x is in $M^{\mathfrak{p}*}$; in fact, if y is the projection of x onto the $M^{\mathfrak{q}*}$ component of $\Pi^{\mathfrak{p}}(M^{\mathfrak{p}*},\ )$ for any $\mathfrak{q}$ different from $\mathfrak{p}$, then y must be 0 since it satisfies both $\lim_h \mathfrak{p}^h y = 0$ and $\lim_h \mathfrak{q}^h y = 0$; hence x is in $M^{\mathfrak{p}*}$ as asserted. This shows that $M^{\mathfrak{p}*}$ is the maximal $\mathfrak{p}$-primary subgroup of $(M_3)^*$. But $(M_3)^* = (M^*)_3$ by part (iv) of this theorem. Therefore $M^{\mathfrak{p}*}$ is the $\mathfrak{p}$-primary part of M^*. This proves (v).

COROLLARY. *A $\mathfrak{p}$-primary $\mathcal{O}$-module is also a topological torsion $\mathcal{O}$-module.*

Proof. Let M be a $\mathfrak{p}$-primary $\mathcal{O}$-module. From the definition of $\mathfrak{p}$-primary it is easily seen that the torsion-free discrete and infinite parts of M are trivial. From part (iii) of the above theorem and the fact that M^* is also $\mathfrak{p}$-primary it follows that the compact divisible part of M is also trivial. Hence M is a topological torsion $\mathcal{O}$-module. The corollary is proved.

4. Modules with Lattice

We say that L is a lattice in the $\mathcal{O}$-module M to mean that L is a discrete submodule of M and that M/L is compact.

Not every $\mathcal{O}$-module contains a lattice. The most classical example of one which does contain a lattice is of course the real vector group of dimension n (considered as a $\mathbf{Z}$-module) which contains $\mathbf{Z}^m$ as a lattice. An example of more recent prominence is the ring of adeles of a global field, k, which contains the subfield k as a lattice.

The main result of this section is a characterization of the $\mathcal{O}$-modules with lattice (Theorem 5). The result seems to be new even for the case of locally compact abelian groups (i.e., $\mathbf{Z}$-modules). We establish Theorem 5 by means of Lemmas 3 and 4. These two lemmas will also find application later in the paper.

First some definitions are needed.

If v is a valuation of k which is finite with respect to $\mathcal{O}$, then $\mathfrak{p} = \{\lambda \in \mathcal{O} \mid v(\lambda) < 1\}$ is a prime ideal in $\mathcal{O}$ and v is called the $\mathfrak{p}$-adic valuation of k. It is well known that for each prime ideal, $\mathfrak{p}$, of $\mathcal{O}$, there is a unique $\mathfrak{p}$-adic valuation of k. We let $k_{\mathfrak{p}}$ be the completion of k with respect to its $\mathfrak{p}$-adic valuation and we let $\mathcal{O}_{\mathfrak{p}}$ be the closure of $\mathcal{O}$ in $k_{\mathfrak{p}}$. It is easily shown that $\mathcal{O}_{\mathfrak{p}}$ is a compact open subring of $k_{\mathfrak{p}}$; in fact for each positive h, the set $\mathfrak{p}^h\mathcal{O}_{\mathfrak{p}}$ is an open subgroup of finite index in $\mathcal{O}_{\mathfrak{p}}$, and as h ranges over all natural numbers, these sets form a neighborhood system of O in $\mathcal{O}_{\mathfrak{p}}$. Multiplication of elements of $k_{\mathfrak{p}}$ by elements of its subring, $\mathcal{O}$, makes $k_{\mathfrak{p}}$ into an $\mathcal{O}$-module and as such $\mathcal{O}_{\mathfrak{p}}$ is a compact open submodule. Note also that $k_{\mathfrak{p}}$ is a $\mathfrak{p}$-primary $\mathcal{O}$-module.

Definition. Let M be an $\mathcal{O}$-module. M is said to have $\mathfrak{p}$-rank 1 if it is isomorphic to $k_{\mathfrak{p}}$, $\mathcal{O}_{\mathfrak{p}}$, $k_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}$, or $\mathcal{O}/\mathfrak{p}^h$ for some natural number h. M is said to have *$\mathfrak{p}$-rank n* where n is a natural number if M is the direct sum of n $\mathcal{O}$-modules each of which has $\mathfrak{p}$-rank 1. Finally M is said to have *$\mathfrak{p}$-rank at most n* if $M = O$ or if M has $\mathfrak{p}$-rank m for some m satisfying $1 \leqslant m \leqslant n$. An $\mathcal{O}$-module satisfying any of these conditions is of course $\mathfrak{p}$-primary.

It turns that $k_{\mathfrak{p}}^* = k_{\mathfrak{p}}$, $\mathcal{O}_{\mathfrak{p}}^* = k_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}}$, and $\mathcal{O}/\mathfrak{p}^h$ is isomorphic to its own dual. Consequently, M^* has $\mathfrak{p}$-rank n if M does. However, these facts will not be used until Section 6.

Recall that $k_\infty = Q_\infty \otimes_Q k$ where Q_∞ is the unique infinite completion of Q. We can now state

Theorem 5. *Let M be an $\mathcal{O}$-module with M_0, M_1, M_2, M_3, and R as in Theorem 0. Then M contains a lattice if and only if*

(i) $R = k_\infty^m$ *for some positive integer m.*

(ii) *There exist closed submodules,* Γ_1 *and* Γ_2, *in* M_3 *such that* $M_3 \supset \Gamma_2 \supset \Gamma_1$, M_3/Γ_2 *is compact,* Γ_1 *is discrete, and* $\Gamma_2/\Gamma_1 = \Pi_{\mathfrak{p}}(\Gamma^{\mathfrak{p}},\ )$ *where* $\Gamma^{\mathfrak{p}}$ *is a* $\mathfrak{p}$*-primary* $\mathcal{O}$*-module of* $\mathfrak{p}$*-rank at most* m *for each prime* $\mathfrak{p}$ *and where the restricted product is taken over all primes in* $\mathcal{O}$.

The proof of Theorem 5 will be based on the following two lemmas.

LEMMA 3. *Let* M *be an* $\mathcal{O}$*-module with* $M = M_2 \oplus R$ *as in Theorem 2. Let* $\mathfrak{p}$ *be a prime ideal in* $\mathcal{O}$. *Let* m *be a natural number. Suppose that for each compact open submodule,* K, *of* M_2, *the* $\mathcal{O}/\mathfrak{p}$ *dimension of* $K/\mathfrak{p}K$ *is at most* m. *Then the* $\mathfrak{p}$*-primary part of* M *has* $\mathfrak{p}$*-rank at most* m.

Proof. Let K be any compact open submodule of M_2. Then $M_1 \supset K \supset M_0$ by the definition of M_1 and M_0.

Then K/M_0 is a compact open submodule of M_3. So by Theorem 3(v), $K/M_0 = \Pi_{\mathfrak{p}} K^{\mathfrak{p}}$, where $K^{\mathfrak{p}}$ is a compact open submodule of $M^{\mathfrak{p}}$ for each prime $\mathfrak{p}$ and where the product is taken over all primes $\mathfrak{p}$. Moreover, it is readily verified that $\mathfrak{p}(K^{\mathfrak{q}}) = K^{\mathfrak{q}}$ for each prime $\mathfrak{q}$ different from $\mathfrak{p}$. Also, $\mathfrak{p}M_0 = M_0$ by the divisibility of M_0. It follows from these observations that $K/\mathfrak{p}K$ is isomorphic to $K^{\mathfrak{p}}/(\mathfrak{p}K^{\mathfrak{p}})$. Therefore, we may with no loss of generality assume that $M = M^{\mathfrak{p}}$. So, we assume that M is a $\mathfrak{p}$-primary $\mathcal{O}$-module and proceed to show that M has $\mathfrak{p}$-rank at most m.

Any $\mathfrak{p}$-primary $\mathcal{O}$-module can be made into a module over the ring, $\mathcal{O}_{\mathfrak{p}}$, in the following way. For each λ in $\mathcal{O}_{\mathfrak{p}}$ write $\lambda = \lim_h \lambda_h$, where the λ_h are in $\mathcal{O}$. Then for each x in M, define $\lambda x = \lim_h \lambda_h x$. That this limit exists in M and is independent of the choice of the sequence $\{\lambda_h\}$ follows readily from the definition of the $\mathfrak{p}$-adic topology in $\mathcal{O}_{\mathfrak{p}}$, the fact that M is $\mathfrak{p}$-primary, and the fact that M is locally compact. It is easily verified that this makes M into an $\mathcal{O}_{\mathfrak{p}}$-module.

Clearly, the $\mathcal{O}_{\mathfrak{p}}$-submodule generated by any nonzero element, x, of M, is either isomorphic to the finite module, $\mathcal{O}/\mathfrak{p}^h$, for some natural number h, or else it is isomorphic as an abstract $\mathcal{O}$-module to $\mathcal{O}_{\mathfrak{p}}$; moreover, in the latter case it is easily seen to be isomorphic to $\mathcal{O}_{\mathfrak{p}}$ as a topological $\mathcal{O}$-module. In particular we see that any finitely generated $\mathcal{O}_{\mathfrak{p}}$-submodule of M will be compact and hence a closed submodule of M.

Let K be any compact open $\mathcal{O}$-submodule of M. Since K is closed in M it is an $\mathcal{O}_{\mathfrak{p}}$-submodule as well. By hypothesis $K/\mathfrak{p}K$ is generated over $\mathcal{O}_{\mathfrak{p}}$ by m elements. We will show from this that K is generated over $\mathcal{O}_{\mathfrak{p}}$ by m elements. Indeed, let s be a set of m elements of K whose images modulo $\mathfrak{p}K$ generate $K/\mathfrak{p}K$ over $\mathcal{O}$. Let N be the $\mathcal{O}_{\mathfrak{p}}$-submodule of K generated by s. Then N is a closed submodule of M by the remarks of the previous paragraph. Hence K/N is a $\mathfrak{p}$-primary $\mathcal{O}$-module. But $\mathfrak{p}(K/N) = K/N$ since $\mathfrak{p}K + N = K$ by our choice of N. Therefore $K/N = O$; otherwise K/N, being a torsion module, would contain a proper open submodule; then the quotient, say B, by that open submodule would be finite, nonzero, $\mathfrak{p}$-primary and satisfy $\mathfrak{p}B = B$ which is absurd.

Any finitely generated $\mathcal{O}_{\mathfrak{p}}$-submodule, say N, of M is generated by m elements. Indeed, N is contained in some compact open submodule, say K, of M. By what we just showed, K is generated over $\mathcal{O}_{\mathfrak{p}}$ by m elements. By the theory of discrete modules over principal ideal rings it follows that N is generated by m elements as asserted.

As an abstract (forgetting topology) $\mathcal{O}_{\mathfrak{p}}$-module, M is isomorphic to the direct sum of r rank 1 $\mathcal{O}_{\mathfrak{p}}$-modules for some positive integer $r \leqslant m$. Indeed, it is known [6, p. 53] that any abstract nontrivial $\mathcal{O}_{\mathfrak{p}}$-module contains a rank-1 direct summand. From this, an induction argument shows that M either has rank r for some $r \leqslant n$, or else M contains a rank $n + 1$ direct summand. But the latter case is impossible because then M would contain an abstract $\mathcal{O}_{\mathfrak{p}}$-submodule generated by $m + 1$ but no fewer elements. Hence M has rank r as an abstract $\mathcal{O}_{\mathfrak{p}}$-module with $r \leqslant m$ which is what we wanted to show.

Thus, there is an abstract $\mathcal{O}_{\mathfrak{p}}$-module, say A, of $\mathfrak{p}$-rank r for some $r \leqslant m$, and an $\mathcal{O}_{\mathfrak{p}}$-linear bijection, $f : A \to M$. Let A be given its usual $\mathfrak{p}$-adic topology. We will show that f is bicontinuous and this will complete the proof of the lemma. Indeed, A contains a compact open $\mathcal{O}_{\mathfrak{p}}$-submodule, say K, which is isomorphic to $(\mathcal{O}_{\mathfrak{p}})^s$ for some $s \leqslant m$. From the fact that M is $\mathfrak{p}$-primary and the definition of the $\mathfrak{p}$-adic topology on $(\mathcal{O}_{\mathfrak{p}})^s$ it is easily shown that f is continuous on K. Then f is continuous on A since K is open in A. But A is the union of countably many compact subsets (since it has finite $\mathfrak{p}$-rank), and it is known that a continuous epimorphism from a group of that sort to any locally compact group is necessarily bicontinuous [13, p. 115]. Therefore, f is bicontinuous. Therefore, M is a $\mathfrak{p}$-primary $\mathcal{O}$-module of $\mathfrak{p}$-rank at most n. This proves the lemma.

DEFINITION. If L is a discrete $\mathcal{O}$-module, define the *torsion-free rank of L over $\mathcal{O}$* to be the $\dim_k(L \otimes_{\mathcal{O}} k)$. Note that this is just the cardinality of a maximal $\mathcal{O}$-linearly independent subset of L. If there is no chance of confusion, we will call it simply the *rank of L*.

LEMMA 4. *Suppose that $M = M_3 \oplus R$ where M_3 is a topological torsion $\mathcal{O}$-module and R is the direct sum of finitely many infinite completions of k. Suppose further that M contains a submodule, L, such that L is discrete torsion free and M/L is compact divisible. Let m be the rank of L. Then*

(i) *m is finite.*

(ii) $R = k_{\infty}{}^{m}$.

(iii) *$M^{\mathfrak{p}}$ has $\mathfrak{p}$-rank at most m for each prime, $\mathfrak{p}$, in $\mathcal{O}$.*

(iv) *The torsion-free rank of $(M/L)^*$ equals m.*

Proof. (1) Suppose that R is the $\mathscr{Z}$-module, $(Q_\infty)^s$, where s is some natural number and that L is a $\mathscr{Z}$-lattice in R. Then, we claim that L is isomorphic to $\mathscr{Z}^s$. Indeed, L generates R over Q_∞; otherwise one would have $R = N_1 \oplus N_2$, where N_2 is the Q_∞-subspace of R generated by L and where N_1 is a nontrivial Q_∞-subspace, and this would be impossible by the compactness of R/L and the noncompactness of N_2. Therefore, L contains a subset, $\{x_1, ..., x_s\}$, which is a basis of R over Q_∞. Let L' be the $\mathscr{Z}$-submodule of R generated by $\{x_1, ..., x_s\}$. It is easily verified that L' is a lattice in R and that L' is isomorphic to $\mathscr{Z}^s$. Then L/L' is finite, since it is a discrete submodule of the compact module R/L'. Moreover, L, being a submodule of R, is torsion free over $\mathscr{Z}$. One can easily see from this and the theory of modules over principal ideal rings that L is itself isomorphic as a $\mathscr{Z}$-module to $\mathscr{Z}^s$, which is what we wanted to prove.

(2) Suppose that R is an $\mathcal{O}$-module which is the direct sum of finitely many infinite completions of k. Suppose further that R contains an $\mathcal{O}$-lattice, L. Let m be the torsion-free rank of L over $\mathcal{O}$. We claim then that m is finite and that $R = (k_\infty)^m$.

To show that L has finite rank, observe that any infinite completion of k is a finite field extension of Q_∞. Hence, R, considered as a $\mathscr{Z}$-module, is just the direct sum of finitely many copies of Q_∞. Hence, by (1), L has finite rank as a $\mathscr{Z}$-module. So certainly L has finite rank as an $\mathcal{O}$-module, as asserted.

Now put $R = \Sigma_i\, k_i^{m(i)}$, where the sum is taken over all of the distinct infinite completions of k (of which there are finitely many) and where for each i, $m(i)$ is a non-negative integer.

We claim that $m(i) \leqslant m$ for each i. Indeed, let f be the canonical projection of R onto its direct summand $k_i^{(m)i}$. Then $f(L)$ generates $k_i^{m(i)}$ over k_i; this follows from the compactness of R/L. It follows that the torsion-free rank of $f(L)$ over $\mathcal{O}$, and hence that of L is at least as great as $m(i)$. Thus $m(i) \leqslant m$ as asserted.

Let $n = [k : Q]$, the degree of k over Q. Then it is well known that $\mathcal{O}$ considered as a $\mathscr{Z}$-module, has rank n. Hence L has rank $m \cdot n$ over $\mathscr{Z}$. Hence, by (1), R has dimension $m \cdot n$ over Q_∞.

But $\Sigma_i k_i{}^m$ also has dimension $m \cdot n$ over Q_∞; since

$$\Sigma_i k_i{}^m = (k_\infty)^m = (Q_\infty \otimes {}_Q k)^m.$$

Thus $R = \Sigma_i k_i^{m(i)}$, where $m(i) \leqslant m$ for all i and where the dimension of R over Q_∞ is equal to the dimension of $\Sigma_i k_i{}^m$ over Q_∞. It follows that $m(i) = m$ for all i and hence $R = (k_\infty)^m$. This proves (2).

(3) Let M_3, R, and L be as in the hypotheses of the lemma. Let K be any compact open submodule of M_3. Let $L' = L \cap (K + R)$. Let L'' be the image of L' under the canonical projection, $f : K + R \to R$. Then, an exercise in group theory shows that L'' is a lattice in R. Moreover, f induces an isomorphism of L' to L'', because L' being torsion-free discrete has trivial intersection with the compact module $K = \ker f$.

Now we can show that m is finite and that $R = (k_\infty)^m$. Indeed, L/L' is a discrete torsion module; this follows from the fact that it is isomorphic to $(K + R + L)/(K + R)$ which is contained in $M(K + R) = M_3/K$ which is a discrete torsion module. Therefore L' has the same torsion-free rank over $\mathcal{O}$ as does L. But $L' \cong L''$. Hence L'' has torsion-free rank m. By (2) it now follows that m is finite and that $R = (k_\infty)^m$ as asserted. So (i) and (ii) have been verified.

(4) It remains to verify (iii). To this end we first show that for any prime $\mathfrak{p}$ of $\mathcal{O}$, $\mathcal{O}/\mathfrak{p}$-$\dim(L'/\mathfrak{p}L') = m$, where L' and m are as in (3). Indeed, from (1) it follows that L'' is finitely generated over $\mathscr{X}$ and hence over $\mathcal{O}$. But $L' \cong L''$. So L' is a finitely generated torsion-free discrete $\mathcal{O}$-module. Hence by the theorem of Steinitz [4], L' is the direct sum of ideals of $\mathcal{O}$. Moreover, there are precisely m ideals in that direct sum decomposition because L' has rank m. But classical ideal theory shows that $\mathcal{O}/\mathfrak{p}$-$\dim(\mathscr{I}/\mathscr{I}\mathfrak{p} = 1$ for any nontrivial ideal, $\mathscr{I}$, of $\mathcal{O}$. It follows that $\mathcal{O}/\mathfrak{p}$-$\dim(L'/\mathfrak{p}L') = m$ as asserted.

(5) Next we show that $\mathcal{O}/\mathfrak{p}$-$\dim(K/\mathfrak{p}K)$ is at most equal to $\mathcal{O}/\mathfrak{p}$-$\dim(L'/\mathfrak{p}L')$ where K and L' are as in (3). Indeed, $K + R + L = M$, since $K + R$ is open in M and M/L, being compact divisible, contains no proper open submodules. Therefore, $(K + R)/L'$ is isomorphic to M/L and is hence divisible. Therefore $\mathfrak{p}(K + R) + L' = K + R$. Therefore, $(K + R)/\mathfrak{p}(K + R)$ is isomorphic to $L'/(L'(K + R))$. But the former of these two modules is isomorphic to $K/\mathfrak{p}K$ and the latter one is a homomorphic image of $L'/\mathfrak{p}L$. Therefore, the $\mathcal{O}/\mathfrak{p}$-dimension of $K/\mathfrak{p}K$ is at most equal to the $\mathcal{O}/\mathfrak{p}$-dimension $L'/\mathfrak{p}L'$ as asserted. This proves (5).

(4), (5), and Lemma 3 together enable us to conclude that $M^{\mathfrak{p}}$ has $\mathfrak{p}$-rank at most m for any prime, $\mathfrak{p}$, in $\mathcal{O}$. This verifies (iii).

(6) If the pair, M, L, satisfy the hypothesis of the lemma, then so does the pair, $M^*, L_\perp$. Hence, $L_\perp$ has torsion-free rank m by part (ii) of the lemma and Theorem 4(i). Hence, $(M/L)^*$, being isomorphic to $L_\perp$, has torsion-free rank m which proves (iv). Lemma 4 is proved.

Proof of Theorem 0. First we show that M contains a lattice if and only if M/M_0 contains a lattice. Indeed, if L is a lattice in M, then L/M_0 is a lattice in M/M_0; this is readily verified by using the compactness of M_0. Conversely, suppose that M/M_0 contains a lattice, L. Then let N be the inverse image of L by the map, $M \to M/M_0$. Then N contains M_0. M_0 is open in N by the discreteness of L and M_0 is divisible by Theorem 3(iv). Hence $N = M_0 + L'$ for some discrete submodule L' of N by Lemma 2. Then, L' is clearly a lattice in M. Thus M contains a lattice if and only if M/M_0 contains a lattice as asserted.

Therefore, we may with no loss of generality assume that $M_0 = 0$. But M contains a lattice if and only if M^* does; in fact if L is a lattice in M, then $L_\perp$ is a lattice in M^*. Therefore we may also assume that $(M^*)_0 = 0$, and hence that $M = M_3 \oplus R$. So suppose that $M = M_3 \oplus R$ and that L is a lattice in M. Let Γ_1 be the torsion submodule of L, let Γ_0 be the intersection of all open submodules of M which contain L, and let Γ_2 be the projection of Γ_0 onto M_3. Then, $\Gamma_0 = \Gamma_2 \oplus R$. Hence, $\Gamma_0/\Gamma_1 = (\Gamma_2/\Gamma_1) \oplus R$. Moreover, L/Γ_1 is a torsion-free discrete submodule of Γ_0/Γ_1 whose quotient, Γ_0/L, is compact and divisible. Then Lemma 4 guarantees that R and Γ_2/Γ_1 satisfy the conditions (i) and (ii) of the theorem. Moreover, by their definitions Γ_1 is discrete and M_3/Γ_2 is compact. Hence conditions (i) and (ii) are satisfied. This proves the "only if" part of Theorem 5.

Conversely, suppose that $M = M_3 \oplus R$ satisfies (i) and (ii) of Theorem 5. We will show that M contains a lattice. Let Γ_1 and Γ_2 be as in (ii). If L is a lattice in $(\Gamma_2/\Gamma_1) \oplus R$, then the inverse image of L by the canonical map, $\Gamma_2 \oplus R \to (\Gamma_2/\Gamma_1) \oplus R$ is a lattice in $M_3 \oplus R$ by the discreteness of Γ_1 and the compactness of M_3/Γ_2. So, we need only show that $(\Gamma_2/\Gamma_1) \oplus R$ contains a lattice.

Let K be a compact open submodule of Γ_2/Γ_1. Then $K = \Pi_p K^p$, where for each p, K^p has p-rank at most m. Thus, for each p, K^p contains a set $\{x_{p1},\dots, x_{pm}\}$ which generates a dense $\mathcal{O}$-submodule of K^p. Moreover, since K^p contains $\mathcal{O}$ as a lattice, it follows that $R = (k_\infty)^m$ contains a set, $\{x_{\infty 1},\dots, x_{\infty m}\}$, which generates over $\mathcal{O}$ a lattice in R. For each $i = 1,\dots, m$, define y_i to be the element of $K \oplus R = (\Pi_p K^p) \oplus R$ whose K^p coordinate is x_{pi} for each prime p and whose R coordinate is $x_{\infty i}$. Let L' be the submodule of $K \oplus R$ generated by $\{y_1,\dots, y_m\}$. An exercise left to the reader shows that L' is discrete and that $(K \oplus R)/L'$ is both compact and divisible. But $(K \oplus R)/L'$ is open in $((\Gamma_2/\Gamma_1) \oplus R)/L'$. Hence, by Lemma 2 the latter of these two modules is the direct sum of the former and a discrete module which we call, L''. Let L be the inverse image of L'' by the canonical map, $(\Gamma_2/\Gamma_1) \oplus R \to ((\Gamma_2/\Gamma_1) \oplus R)/L'$. Then L is a lattice in $(\Gamma_2/\Gamma_1) \oplus R$. We have shown that $(\Gamma_2/\Gamma_1) \oplus R$ has a lattice. The proof of Theorem 5 is complete.

5. Adele Rings

Let $\mathcal{O}$ be a ring of integers in the global field, k. Let

$$A_k = \Pi_p(k_p, \mathcal{O}_p) \oplus \Sigma_v k_v$$

where v indexes the infinite completions of k. Then A_k with multiplication defined coordinatewise forms a locally compact ring, called the ring of adeles of k.

In this section we use the theory of $\mathcal{O}$-modules to obtain several characterizations of A_k. Some of these characterizations are improvements of known results, and some of them are entirely new.

For each λ in k, let $j(\lambda)$ be the element of A_k all of whose coordinates are λ. Then j identifies k and also $\mathcal{O}$ with subrings of A_k. A_k is of course an $\mathcal{O}$-module where scalar multiplication is defined coordinatewise or equivalently as multiplication by elements of the subring $\mathcal{O}$. It turns out [17] that there is an $\mathcal{O}$-module isomorphism of A_k to $A_k{}^*$ which carries k onto $k_\perp$. Consequently, A_k/k is isomorphic as an $\mathcal{O}$-module to k^*. In particular, we see that k is a lattice in A_k. These properties of A_k play an important role in number theory and also in the characterizations that follow.

THEOREM 6. *Let $\mathcal{O}$ be a ring of integers in the global field, k, and let j be as above. Then,*

(i) *A_k considered as an $\mathcal{O}$-module has trivial torsion-free discrete and compact divisible parts.*

(ii) *If M is an $\mathcal{O}$-module with trivial torsion-free discrete and compact divisible parts and if $i : k \to M$, then there exists a unique $\mathcal{O}$-linear map, $f : A_k \to M$ such that $f \circ j = i$.*

(iii) *Properties* (i) *and* (ii) *characterize A_k.*

Proof (i) follows immediately from the definition of A_k.

Now we prove (ii). We may write $M = \Pi_{p}(M^{p},\ ) \oplus \Sigma_v k_v^{n(v)}$ since M has trivial torsion-free discrete and compact divisible parts. Let i_p be the map i followed by projection onto M^p for each prime p, and let i_v be the map i followed by projection onto $k_v^{n(v)}$. Then by definition of k_p and of "p-primary" it follows that i_p extends uniquely to a map $i_p' : k_p \to M^p$. Also that i_v' extends uniquely to $i_v' : k \to k_v^{n(v)}$. Then the maps i_p' and i_v' extend uniquely to $f : \Pi(k_p, \mathcal{O}_p) \oplus \Sigma k_v \to \Pi_p(M_p,\ ) \oplus \Sigma k_v^{n(v)}$. Then clearly $f \circ j = i$. Moreover, f is the only such map, since it is determined by its values on each of the k_p and k_v components of A_k. This proves (ii).

As for (iii), suppose that B is an $\mathcal{O}$-module and that $h : k \to B$ and that B, h satisfy the properties (i) and (iii) above the same as A_k, j do. Then there is an $f : A_k \to B$ such that $f \circ j = 1$ and there is $g : B \to A_k$ such that $g \circ h = j$. Then $g \circ f$ is an endomorphism of A_k which is the identity on $j(k)$. But the identity on $j(k)$ extends uniquely to an endomorphism of A_k by (iii). Hence $g \circ f = 1$. Similarly $f \circ g = 1$. Hence B is isomorphic to A_k which proves (iii). Theorem 6 is proved.

COROLLARY. *Let k be a global field. Suppose that B is a topological ring with no proper open ideals (left or right) and that $i : k \to B$ is a ring homomorphism with $i(1)$ equal to the identity in B. Then, there exists a unique ring homomorphism $f : A_k \to B$ such that $f \circ j = i$. Moreover, this property characterizes the ring of adeles, A_k.*

Proof. Multiplication on the left by elements of $i(k)$ makes B into a left k-module, and in particular a left $\mathcal{O}$-module. Considering B as a left $\mathcal{O}$-module, let $B_1 + R$ and B_0 be as usual. Then $B_1 + R$ and B_0 are right ideals, since they are uniquely determined left $\mathcal{O}$-modules. Then $B_1 + R = B$ since B contains no proper open right ideals. Also, $B_0 = 0$; in fact, a duality argument shows that if a ring has no proper open left ideals then it has no nontrivial compact right ideals (cf. [4, Proposition 1.1]). So, the hypotheses of Theorem 6 are satisfied. Hence there is an $\mathcal{O}$-linear map, $f : A_k \to B$, such that $f \circ j = i$. The proof will be completed if we can show that f is actually a ring homomorphism.

f is left k-linear; this follows easily from A being a torsion-free left $\mathcal{O}$-module (since it is a k-module) and from f being left $\mathcal{O}$-linear.

Fix y in A_k, and consider the two maps, each from A_k to A, defined by $x \to f(x) \cdot f(y)$ and $x \to f(x \cdot y)$ respectively. These two maps are equal on $j(k)$ since f is left k-linear. Hence, by the uniqueness part of the Theorem 6(ii), these two maps are equal on all of A_k. Thus, $f(xy) = f(x)f(y)$ for all x in A_k. Since y was arbitrary this shows that f is a ring homomorphism. The corollary is proved.

If N and N' are $\mathcal{O}$-modules then we say that $i : N \to M$ is an *extension of N by N'* to mean that M is an $\mathcal{O}$-module, that i is a map identifying N with a closed submodule of M, and that $M/i(N) \cong N'$. We say that two extensions of N by N', $i : N \to M$, are *equivalent* if there is an isomorphism $g : M \cong M'$ such that $g \circ i = i'$. An extension, $i : N \to M$, of N by N' is called *split* if $i(N)$ is a direct summand of M and is called *nonsplit* otherwise.

Among the topological modules over discrete fields the adele rings are characterized as follows.

THEOREM 7. *Suppose that k is a discrete field which is not a purely algebraic extension of a finite field and that $i : k \to A$ is a nonsplit extension of k by k^* in the category of k-modules. Then k is a global field and the two extensions, $i : k \to A$, and $j : k \to A_k$, are equivalent.*

Proof. k contains a subring, say $\mathscr{Z}$, which is either the ring of rational integers or else the ring of polynomials in one variable with finite constant field, since k is assumed to be not a purely algebraic extension of a finite field.

Considering A as a $\mathscr{Z}$-module, let A_0 and $A_1 + R$ be as in Theorem 3.

Then A_0 and $A_1 + R$ are k-submodules of A, because they are uniquely determined $\mathscr{Z}$-modules and because k is commutative.

We claim that $A_1 + R = A$. Indeed, $i(k) \subset A_1 + R$; otherwise $A = i(k) \oplus M$ where M is a k-submodule maximal with respect to the properties of containing $A_1 + R$ and having trivial intersection with $i(k)$, and this would contradict the nonsplit hypothesis. But $A/i(k)$ contains no proper open k-submodules, because it is isomorphic to k^* whose dual, k, contains no compact k-submodules. It follows that $A_1 + R = A$ as asserted.

The same argument applied to A^* along with Theorem 4(ii) shows that $A_0 = 0$.

Hence, by Lemma 4, k has finite rank over $\mathscr{Z}$. It follows (by choosing a different subring, $\mathscr{Z}$, if necessary to guarantee separability) that k is a global field.

The first conclusion is proven. It remains to show that the two extensions are equivalent.

Let $\mathscr{O}$ be a ring of integers of k. Then, considering A as an $\mathscr{O}$-module, we still have $A_1 + R = A$ and $A_0 = 0$. Hence, by Theorem 6, there exists an $\mathscr{O}$-linear map, $f: A_k \to A$, such that $f \circ j = i$. Then, f induces a map, $f: A_k/j(k) \to A/i(k)$. But each of these two modules is isomorphic to k^*. A duality argument shows that a nonzero $\mathscr{O}$-linear map of k^* to itself is necessarily an isomorphism. f' cannot be zero, since then A_k would split into the direct sum of k and the kernel of f which is absurd. Hence f' is an isomorphism. It follows easily from this that f is an isomorphism. Thus the two extensions are equivalent. The theorem is proved.

Remark. Theorem 7 says that if k is a global field, then $j: k \to A_k$ is the unique nonsplit extension of k by k^* in the category of k-modules. Actually it is the unique nonsplit extension in the category of $\mathscr{O}$-modules as well. Indeed, if $i: k \to A$ is an extension of k by k^* in the category of $\mathscr{O}$-modules, then A is a torsion-free and divisible $\mathscr{O}$-module since both k and k^* are, and from this it follows that A is actually a k-module. So, any $\mathscr{O}$-module extension of k by k^* is a k-module extension as well and hence equivalent to $j: k \to A_k$.

The next theorem shows that, other than the obvious exceptions, the adele rings are the only topological rings which contain a subfield as lattice.

THEOREM 8. *Suppose that A is a topological ring, that k is a subfield containing the identity of A and that k is a lattice in A. Then, one of the following cases must hold:*

(i) *k is finite and A is a compact ring.*

(ii) *$A = k \oplus M$ where M is a compact two-sided k-module and $M \cdot M = 0$.*

(iii) *k is a global field and A is the ring of adeles of k.*

Proof. (1) If A contains an open left or right ideal, then A satisfies either (i) or (ii). Indeed, suppose that M is a proper open left ideal in A. We suppose further that k is infinite [otherwise A satisfies (i) and we are done]. Then, $A/(k + M)$, being both compact and discrete, is finite. Then $A/(k + M) = 0$, since it is a finite vector space over an infinite field. Then $A = k \oplus M$ as a k-module; in fact $k \cap M = 0$, since k is a field and M is an ideal. It remains to show that M is right k-invariant and that $M \cdot M \neq 0$. M is compact (since it is isomorphic to A/k) and k is assumed infinite; from this it follows that any open left ideal of A contains M. Hence, M is the unique open left ideal of A. It follows from this that M is a right ideal as well, and in particular M is a two-sided k-module. Moreover, by considering M as a compact right A-module, M^* becomes a discrete left A-module in the obvious way. The annihilating left ideal in A of an element of M^* is open in A (by the discreteness of M^* and a continuity argument), and hence must contain M. Thus $M \cdot M^* = 0$, and hence $M \cdot M = 0$. We have shown that A satisfies (i) or (ii). A similar argument holds for open right ideals.

(2) Suppose k an infinite algebraic extension of a finite field. We show that A contains a proper open ideal. Indeed, suppose A has no proper open ideals. It follows by a duality argument that A has no nonzero compact ideals [cf. 4, Prop. 1.1].

We claim that k is in the center of A. Let λ be any element of k. λ is algebraic ove the prime field in k, since k is an algebraic extension of a finite field. Let $m(x)$ be the minimal polynomial of λ over the prime field. Left multiplication by elements of k makes A into a k-vector space. $x \to x\lambda$ is a linear transformation of that vector space, whose minimal polynomial divides $m(x)$. From primary decomposition theorem of linear algebra, $A = A_1 \oplus A_2$ where A_1 is the eigenspace, $\{x \in A : x\lambda = \lambda x\}$, of the linear transformation, $x \to x\lambda$, and A_2 a subspace of A. The direct sum is topological since the projection maps, being polynomials in λ, are continuous. A_2 is compact, because $k \subset A_1$ and A/k compact. $A = k + C$ for some compact $C \subset A$, since A/k compact. Hence $AA_2 = kA_2 + CA_2 = A_2 + CA_2$ is compact left ideal in A. Hence, $AA_2 = 0$. Hence, $A = A_1$. Hence, λ in center of A. Hence, k contained in center of A as claimed.

Let M be a closed maximal left ideal in A; the existence of such M in any locally compact ring was proved by Kaplansky [8, Theorem 11]. M is two sided ideal in the ring $k + M$, since k is in center of A. Let A' be closure of $k + M$ in A. Then A'/M is locally compact field with subfield k. From their classification, a nondiscrete locally compact field cannot contain an infinite algebraic extension of a finite field. Hence A'/M discrete. Hence M open in $k + M$. This and compactness of A/k imply that the ideal M is compact. Hence

$M = 0$. Hence A is a field. By argument just applied to A'/M, A is discrete. So 0 is open ideal in A. (2) is proved.

(3) If A contains no proper open ideals, then k is a global field and A is its ring of adeles. Indeed, suppose that A contains no proper open ideals. Then k is not an infinite algebraic extension of a finite field by (2), and k is not finite, since then A would be compact, and it is known that any compact ring contains open ideals. Hence, k contains a subring, say $\mathscr{Z}$, which is either the ring of rational integers or else the ring of polynomials in one variable over finite constant field. Considering A as a left $\mathscr{Z}$-module, let $A_1 + R$ and A_0 be as in Theorem 3. Then $A_1 + R$ and A_0 are right ideals in A since they are uniquely determined left k-modules. Then $A_1 + R = A$ since A contains no proper open ideals. Also $A_0 = 0$; in fact a duality argument shows that if a ring contains no proper open left ideals, then it also contains no nontrivial compact ideals [4, Proposition 1.1]. Moreover, A/k is divisible as a $\mathscr{Z}$-module, since it is a module over the field k containing $\mathscr{Z}$. So, the hypotheses of Lemma 4 are satisfied. Therefore, by Lemma 4, k has finite rank over $\mathscr{Z}$. It follows (by choosing a different subring, $\mathscr{Z}$, if necessary to guarantee separability) that k is a global field.

It remains to show that A is the ring of adeles of k. By the corollary to Theorem 6, there exists a ring homomorphism, $f : A_k \to A$ which restricts to the identity on k. But $A/k \simeq k^*$; in fact, the k-module, $(A/k)^*$, has the same $\mathscr{Z}$-rank as k by Lemma 4, and is hence isomorphic to k, and hence $A/k \simeq k^*$ as asserted. Then by the same argument used in the end of the proof of Theorem 7, we see that f is actually an isomorphism. Thus A is isomorphic to the ring of adeles of k. By virtue of (1) and (3) together, Theorem 8 is proved.

Remark. Theorems 6 and 7 are new results. The corollary to Theorem 6 strengthens a theorem of Goldman and Sah [3, Theorem 5.1]. Theorem 8 strengthens a theorem of Iwasawa [5, p. 339]. For characterizations of $k_{\mathfrak{p}}$, $\mathcal{O}_{\mathfrak{p}}$, k_∞, and $k_\infty/\mathcal{O}$ in the special case of Z-modules the reader is referred to [15]; some of these results generalize easily to $\mathcal{O}$-modules.

6. Applications and Examples

The following theorem will be needed in a subsequent paper [11]. Its proof is an application of Lemmas 3 and 4.

Theorem 9. *Let M be an $\mathcal{O}$-module with closed submodule, N. Suppose that $N^{\mathfrak{p}}$ and $(M/N)^{\mathfrak{p}}$ both have finite $\mathfrak{p}$-rank for some prime, $\mathfrak{p}$, in $\mathcal{O}$. Then $M^{\mathfrak{p}}$ also has finite $\mathfrak{p}$-rank.*

Proof. Suppose that $N^{\mathfrak{p}}$ has $\mathfrak{p}$-rank r, $(M/N)^{\mathfrak{p}}$ has $\mathfrak{p}$-rank s and that $m = \min\{m(i)\}_i$, where $R = \Sigma_i k_i^{m(i)}$ is the infinite part of M. We will show that M has $\mathfrak{p}$-rank at most $r + s + m$.

Case I. Suppose that $(M/N)_0 = 0$. We claim that $M^{\mathfrak{p}}$ has $\mathfrak{p}$-rank at most $r + s$. Indeed, let f be the canonical map $f : M \to M/N$. Then $f(M_0) = 0$, since $(M/N)_0 = 0$. So f induces a map $f' : M/M_0 \to M/N$. Let K be an arbitrary compact open submodule of M_1/M_0. Let $K^{\mathfrak{p}}$ be the $\mathfrak{p}$-primary part of K/M_0. The kernel of f' is a homomorphic image of N, and hence its $\mathfrak{p}$-primary part has $\mathfrak{p}$-rank at most r. It follows that the intersection of $K^{\mathfrak{p}}$ with the kernel of f' is generated over $\mathcal{O}_{\mathfrak{p}}$ by r elements. Moreover, $f'(K^{\mathfrak{p}})$ is generated over $\mathcal{O}_{\mathfrak{p}}$ by s elements since it is a compact submodule of (M/N). Therefore $K^{\mathfrak{p}}$ is generated over $\mathcal{O}_{\mathfrak{p}}$ by $r + s$ elements. Therefore, the $\mathcal{O}/_{\mathfrak{p}}$-dim$(K^{\mathfrak{p}}/_{\mathfrak{p}}K^{\mathfrak{p}}) \leqslant r + s$. Therefore, by Lemma 3, the $\mathfrak{p}$-primary part, $M^{\mathfrak{p}}$, of M_1/M_0 has $\mathfrak{p}$-rank at most $r + s$. But $M^{\mathfrak{p}}$ is also the $\mathfrak{p}$-primary part of M. Thus the $\mathfrak{p}$-primary part of M has $\mathfrak{p}$-rank at most $r + s$ which is what we wanted to prove.

Case II. Suppose that $N = N_1 + R$ (i.e., N has trivial torsion-free discrete part). Then the $\mathfrak{p}$-rank of M is at most $r + s$. This follows from Case I and a duality argument.

Case III. M and N arbitrary. Let A be the inverse image of $(M/N)_0$ under the map, $M \to M/N$. Let $B = N_1 + R$, where $N = N_2 + R$ in the notation of Theorem 3. Then $M \supset A \supset N \supset B$. The $\mathfrak{p}$-primary part of M/A, being equal to the $\mathfrak{p}$-primary part of M/N, has $\mathfrak{p}$-rank s, and similarly the $\mathfrak{p}$-primary part of B has $\mathfrak{p}$-rank r. N/B is a lattice in A/B satisfying the hypothesis of Lemma 4. It follows by that lemma that the $\mathfrak{p}$-primary part of A/B has $\mathfrak{p}$-rank at most m. Applying Case I to the pair M/B, A/B, one concludes that M/B has $\mathfrak{p}$-rank at most $m + s$. Then applying Case II to the pair, M, B, one concludes that the $\mathfrak{p}$-primary part of M has $\mathfrak{p}$-rank at most $r + m + s$. The theorem is proved.

The rest of this section will be devoted to examples.

EXAMPLE 1. Theorems 2 and 5 show the distinctly different roles played by the infinite and finite completions of k in the structure theory of $\mathcal{O}$-modules. This example will illustrate further this distinction. An infinite completion of k is direct summand of any $\mathcal{O}$-module containing it; this is easily verified using Theorem 2 and the fact that an infinite completion of k contains no proper open nor nontrivial compact $\mathcal{O}$-submodules. We give an example to show that for any finite completion, say $k_{\mathfrak{p}}$, of k, there is an $\mathcal{O}$-module, say M, such that $k_{\mathfrak{p}}$ is contained in M but is not a direct summand of M.

Let $A = \prod_{h=1}^{\infty} \mathcal{O}/_{\mathfrak{p}}{}^h$. Then A is a compact $\mathfrak{p}$-primary $\mathcal{O}$-module. Let x

be an element of infinite order in A; for instance x could be the element whose $\mathcal{O}/_{\mathfrak{p}}{}^h$ coordinate is 1 for each h. Let B be the closure of the $\mathcal{O}$-submodule of A generated by x. Then B is isomorphic to $\mathcal{O}_{\mathfrak{p}}$. We "attach" this $\mathcal{O}_{\mathfrak{p}}$ to the submodule, $\mathcal{O}_{\mathfrak{p}}$, of $k_{\mathfrak{p}}$ and in this way form a module which contains $k_{\mathfrak{p}}$ but not as a direct summand. More formally, let N be the closure of the $\mathcal{O}$-submodule of $k_{\mathfrak{p}} \oplus A$ generated by $(1, x)$, and let $M = (k_{\mathfrak{p}} \oplus A)/N$. Then the map, $f : k_{\mathfrak{p}} \oplus A \to (k_{\mathfrak{p}} \oplus A)/N$, maps $k_{\mathfrak{p}}$ isomorphically onto a closed submodule of M.

But $f(k_{\mathfrak{p}})$ is not a direct summand of M. Indeed, suppose $M = f(k_{\mathfrak{p}}) \oplus M'$. Then all of the elements of finite order are contained in M'. But $f(1)$ is a cluster point of such elements since it equals $f(x)$. Thus $f(1)$ is in M' which is absurd.

EXAMPLE 2. We give an example of a module with lattice and an infinite number of examples of modules without lattice. Let M be as in Example 1. Then $M + k_\infty$ contains a lattice. Take $\Gamma_1 = 0$, $\Gamma_2 = k_{\mathfrak{p}} \subset M$. Then $M/k_{\mathfrak{p}} \simeq A/B$ compact. So $M + k_\infty$ contains a lattice.

Now take $M = \prod_{h=1}^{\infty} (\mathcal{O}/\mathfrak{p}^{2h}, {}_{\mathfrak{p}}{}^h\mathcal{O}/\mathfrak{p}^{2h})$. Let Γ_1 be any discrete subgroup of M, then $\Gamma_1 \cap M_{\mathfrak{p}}$ being both discrete and compact is finite. From this and the fact that every element of $M_{\mathfrak{p}}$ has finite height, it follows that Γ_1 is finite. But M is isomorphic to its own dual. Hence, if M/Γ_2 is compact for some closed submodule Γ_2 in M then Γ_2 has finite index in M. But if Γ_2 has finite index in M and Γ_1 is finite, then Γ_2/Γ_1 contains $\prod_{h=n}^{\infty} (\mathcal{O}/\mathfrak{p}^{2h}, {}_{\mathfrak{p}}{}^h\mathcal{O}/\mathfrak{p}^{2h})$ for some sufficiently large n. But this module does not have finite $\mathfrak{p}$-rank, because for instance its set of points of order $\mathfrak{p}$ is infinite. Therefore M fails to satisfy requirement (ii) of Theorem 5. Therefore $M \oplus (k_\infty)^m$ fails to contain a lattice for every natural number m.

EXAMPLE 3. If $M^{\mathfrak{p}}$ is a $\mathfrak{p}$-primary $\mathcal{O}$-module with compact open submodule, $K^{\mathfrak{p}}$ for each prime $\mathfrak{p}$, then $\Pi_{\mathfrak{p}}(M^{\mathfrak{p}}, K^{\mathfrak{p}})$ is a topological torsion module, and conversely, by Theorem 3(v) every topological torsion module can be obtained in this way. The point we wish to make here is that if the $M^{\mathfrak{p}}$'s are given, then different choices for the $K^{\mathfrak{p}}$'s can result in enormously different product modules.

The simplest example of this is that if all $M^{\mathfrak{p}}$'s are finite, then $\Pi_{\mathfrak{p}}(M^{\mathfrak{p}}, M^{\mathfrak{p}})$ is compact whereas $\Pi_{\mathfrak{p}}(M^{\mathfrak{p}}, 0)$ is discrete.

For a second example, we let $M^{\mathfrak{p}} = (\mathcal{O}/\mathfrak{p}) \oplus k_{\mathfrak{p}}$ for each prime $\mathfrak{p}$. If for each prime, $\mathfrak{p}$, we choose $K^{\mathfrak{p}}$ to be the $\mathcal{O}_{\mathfrak{p}}$-submodule of $M^{\mathfrak{p}}$ generated by $(0, 1)$ in $(\mathcal{O}/\mathfrak{p}) \oplus k_{\mathfrak{p}}$, then $M_3 = \Pi_{\mathfrak{p}}(M^{\mathfrak{p}}, K^{\mathfrak{p}})$ decomposes into the direct sum $\Pi_{\mathfrak{p}}(\mathcal{O}/\mathfrak{p}, 0) \oplus \Pi_{\mathfrak{p}}(k_{\mathfrak{p}}, \mathcal{O}_{\mathfrak{p}})$. In particular, we see in this case that $M_3 \oplus k_\infty$ contains a lattice by Theorem 5. However, if for each $\mathfrak{p}$, we choose $K^{\mathfrak{p}}$ to be the $\mathcal{O}_{\mathfrak{p}}$-submodule of M generated by $(1, 1)$, then $M_3 = \Pi_{\mathfrak{p}}(M^{\mathfrak{p}}, K^{\mathfrak{p}})$ no longer splits into the restricted product of rank-1 modules. Moreover, unlike the previous case, $M_3 \oplus k_\infty$ will not contain a lattice (although $M_3 \oplus k_\infty{}^2$ will contain a lattice). We leave this to the reader to verify.

7. Duality Reconsidered

Let $\mathscr{R}$ be any commutative ring. In Section 2 we defined the duality functor, $*$, on the category of $\mathscr{R}$-modules. In this section we show that $*$ is representable. For the case in which $\mathscr{R}$ is a Dedekind Domain we give explicitly all of the representable invertible functors on the category of $\mathscr{R}$-modules. They turn out to be in a two (one covariant and one contravariant) to one correspondence with the ideal classes of $\mathscr{R}$.

Let $\mathscr{R}$ be any commutative ring until explicitly stated otherwise. Assume throughout that $\mathscr{R}$ has the discrete topology.

Definition. Two contravariant (resp. covariant) functors, $\#$ and $'$, on the category of $\mathscr{R}$-modules are said to be *equivalent* if for each $\mathscr{R}$-module, M, there is an isomorphism, $\phi_M : M' \simeq M^{\#}$, such that if M and N are $\mathscr{R}$-modules and $f : M \to N$, then $f^{\#}\phi_N = \phi_M f'$ (resp. $f^{\#}\phi_M = \phi_N f'$).

By a *topological module over* $\mathscr{R}$ we mean an object which is a topological space and a module over $\mathscr{R}$ with the module operations continuous. It may fail to be an $\mathscr{R}$-module for lack of local compactness.

Definition. Let A be any $\mathscr{R}$-module. Define the contravariant (resp. covariant) functor, $\mathscr{H}(\ , A)$, [resp. $\mathscr{H}(A,\ )$] from the category of $\mathscr{R}$-modules to the category of topological modules over $\mathscr{R}$ as follows. For each $\mathscr{R}$-module M, define $\mathscr{H}(M, A)$ [resp. $\mathscr{H}(A, M)$] to be the set of all $\mathscr{R}$-homomorphisms of M into A (resp. of A into M). With pointwise addition and scalar multiplication and the compact open topology, $\mathscr{H}(M, A)$, [resp. $\mathscr{H}(A, M)$] becomes a topological module over $\mathscr{R}$. If M and N are any $\mathscr{R}$-modules and $f : M \to N$, define $\mathscr{H}(f, A)$ to be the map, $f' : \mathscr{H}(N, A) \to \mathscr{H}(M, A)$, defined by $f'(g) = g \circ f$ for every g in $\mathscr{H}(N, A)$ [resp. define $\mathscr{H}(A, f)$ to be the map $f' : \mathscr{H}(M, A) \to \mathscr{H}(N, A)$ defined by $f'(g) = f \circ g$ for every g in $\mathscr{H}(M, A)$]. This makes $\mathscr{H}(\ , A)$ and $\mathscr{H}(A,\ )$ into functors from the category of $\mathscr{R}$-modules to the category of topological modules over $\mathscr{R}$.

Definition. We say that a functor from the category of $\mathscr{R}$-modules to itself is *representable* if it is equivalent to $\mathscr{H}(A,\ )$ or $\mathscr{H}(\ , A)$ for some $\mathscr{R}$-module, A. [In particular, then, $\mathscr{H}(A, M)$ or $\mathscr{H}(M, A)$ will have to be locally compact for all M.]

THEOREM 10. *Let $\mathscr{R}$ be any commutative ring. Let $*$ be as in Section 2. Then, $*$ is representable. In fact, it is equivalent to $\mathscr{H}(\ ,\mathscr{R}^*)$. In particular, if $\mathscr{R} = \mathscr{O}$ is a ring of integers of the global field, k, then $*$ is equivalent to $\mathscr{H}(\ , k_\infty/\mathscr{O})$.*

Proof. We wish to define an isomorphism, $\phi_M : M^* \simeq \mathscr{H}(M, \mathscr{R}^*)$. For each y in M^*, x in M, and λ in $\mathscr{R}$, put $((\phi_M(y))(x))(\lambda) = y(\lambda x)$. This makes $(\phi_M(y))(x)$ an element of $\mathscr{R}^*$; hence it makes $\phi_M(y)$ a map of M to $\mathscr{R}^*$. Several easy computations show that ϕ_M is an isomorphism of M^* to $\mathscr{H}(M, \mathscr{R}^*)$ and that the ϕ_M commute with morphisms in the category of $\mathscr{R}$-modules. This proves the first two statements of the theorem. The last follows from Lemma 1.

DEFINITION. A functor, say Φ, on the category of $\mathscr{R}$-modules is said to be *invertible* if there exists another functor, Φ^{-1}, on the category of $\mathscr{R}$-modules such that $\Phi \circ \Phi^{-1}$ and $\Phi^{-1} \circ \Phi$ are both equivalent to the identity functor.

DEFINITION. Let A and M be $\mathscr{R}$-modules. If $\mathscr{H}(M, A)$ is an $\mathscr{R}$-module (i.e., locally compact), then there is a canonical map, e, of M into its second "A-dual," $\mathscr{H}(\mathscr{H}(M, A), A)$, defined by $(e(x))(y) = y(x)$. Let A be an $\mathscr{R}$-module. We say that $\mathscr{H}(\ , A)$ is a *duality functor* on the category of $\mathscr{R}$-modules to mean that for every $\mathscr{R}$-module, M, $\mathscr{H}(M, A)$ is locally compact and the canonical map, e, is an isomorphism of $\mathscr{R}$-modules. A duality functor is obviously invertible and is its own inverse.

Pontryagin showed that $\mathscr{H}(\ , T)$ is the only duality functor on the category of $\mathbf{Z}$-modules [13, p. 276]. This is a special case of the more general situation described by Theorem 11 below.

Recall that in a Dedekind domain the set of fractional ideals form a group under multiplication and that this group modulo the group of principal fractional ideals is called the ideal class group.

THEOREM 11. *Let $\mathscr{R}$ be a Dedekind domain. Then*

(i) *$\mathscr{H}(\mathscr{I},\ ) \circ \mathscr{H}(\mathscr{J},\ )$ is equivalent to $\mathscr{H}(\mathscr{I}\mathscr{J},\ )$ for any two ideals, $\mathscr{I}$ and $\mathscr{J}$, of $\mathscr{R}$.*

(ii) *$\mathscr{H}(\mathscr{I},\ ) \circ *$ is equivalent to $\mathscr{H}(\ , \mathscr{I}^*)$ for any ideal $\mathscr{I}$ of $\mathscr{R}$.*

(iii) *$\mathscr{H}(A,\ )$ is an invertible functor on the category of $\mathscr{R}$-modules if and only if A is a nonzero ideal of $\mathscr{R}$. If this is the case then its inverse is $\mathscr{H}(A^{-1},\ )$.*

(iv) *$\mathscr{H}(\ , A)$ is an invertible functor on the category of $\mathscr{R}$-modules if and only if A^* is a nonzero ideal of $\mathscr{R}$. If this is the case, then it is a duality functor.*

(v) *The set of invertible representable covariant functors on the category of $\mathscr{R}$-modules with the operation of composition modulo functor equivalence form a group which is isomorphic to the ideal class group of $\mathscr{R}$.*

Proof. (i) For abstract $\mathscr{R}$-modules there is the usual equivalence of $\mathscr{H}(\mathscr{I},\ )\circ\mathscr{H}(\mathscr{J},\ )$ to $\mathscr{H}(\mathscr{I}\otimes\mathscr{J},\ )$ where the tensor product is taken over $\mathscr{R}$. Elementary considerations show that the isomorphisms defining this equivalence are bicontinuous. Moreover, $\mathscr{I}\otimes\mathscr{J}=\mathscr{I}\mathscr{J}$ for any two ideals, $\mathscr{I}$ and $\mathscr{J}$ in a Dedekind domain. Statement (i) follows.

(ii) For each $\mathscr{R}$-module, M, there is an isomorphism,

$$\mathscr{H}(M,\mathscr{I}^*)\to\mathscr{H}(\mathscr{I},M^*),$$

defined by $f\to f^*$. Elementary considerations show that this map is bicontinuous and commutes with the appropriate $\mathscr{R}$-morphisms. Thus it defines a functorial equivalence of $\mathscr{H}(\ ,\mathscr{I}^*)$ to $\mathscr{H}(\mathscr{I},\ )\circ *$. This proves (ii).

(iii) If $\mathscr{I}$ is any nonzero ideal of $\mathscr{R}$, then $\mathscr{H}(\mathscr{I},\ )$ is invertible; in fact $\mathscr{H}(\mathscr{I}^{-1},\ )$ is its inverse by (i).

Conversely, suppose that $\mathscr{H}(A,\ )$ is an invertible functor on the category of $\mathscr{R}$-modules with inverse, Φ. We show that A is an ideal of $\mathscr{R}$. Indeed, $\mathscr{H}(A,\mathscr{R})$ must be nonzero, since $\mathscr{H}(A,\ )$ is invertible. Hence, there is a nontrivial map, $f: A\to\mathscr{R}$. But, for abstract modules over Dedekind domains any such map has a cross section [7, Lemma 2]. Hence, $A=K\oplus\mathscr{I}$ where $\mathscr{I}$ is an ideal of $\mathscr{R}$ and K is the kernel of f. The direct sum is topological, since K is open and $\mathscr{I}$ is discrete. Therefore,

$$\mathscr{H}(A,\Phi(\mathscr{R}^*))=\mathscr{H}(K,\Phi(\mathscr{R}^*))\oplus\mathscr{H}(\mathscr{I},\Phi(\mathscr{R}^*)).$$

But $\mathscr{H}(A,\Phi(\mathscr{R}^*))=\mathscr{R}^*$ which is indecomposable, and the second term of the above decomposition is nonzero, since $\mathscr{H}(\mathscr{I},\ )$ is invertible. Hence

$$(1)\ \mathscr{H}(K,\Phi(\mathscr{R}^*))=0 \quad\text{and}\quad (2)\ \mathscr{H}(\mathscr{I},\Phi(\mathscr{R}^*))=\mathscr{R}^*.$$

Applying $\mathscr{H}(\mathscr{I}^{-1},\ )$ to (2) yields $\Phi(\mathscr{R}^*)=(\mathscr{I}^{-1})^*$. By plugging this value for $\Phi(\mathscr{R}^*)$ into (1) and applying part (ii) of this theorem, one can show that $K=0$. Hence, $A=\mathscr{I}$ is an ideal in $\mathscr{R}$ which is what we wanted to show. This proves (iii).

(iv) $\mathscr{H}(\ ,A)$ is equivalent to $\mathscr{H}(A^*,\ )\circ *$ for any $\mathscr{R}$-module, A, by the same argument that was used to prove (ii). From this and (iii), one sees that $\mathscr{H}(\ ,A)$ is invertible if and only if A^* is an ideal in $\mathscr{R}$. This proves the first statement of (iv).

To prove the second statement of (iv), let $\mathscr{I}$ be any nonzero ideal of $\mathscr{R}$ and let e be the canonical map of M into $\mathscr{H}(\mathscr{H}(M,\mathscr{I}^*),\mathscr{I}^*)$. Identify in the obvious way M and $\mathscr{I}^*$ with $\mathscr{H}(\mathscr{R},M)$ and $\mathscr{H}(\mathscr{R},\mathscr{I}^*)$ respectively. Then e becomes identified with the map, $e':\mathscr{H}(\mathscr{R},M)\to\mathscr{H}(\mathscr{H}(M,\mathscr{I}^*),\mathscr{H}(\mathscr{R},\mathscr{I}^*))$ which sends f to $\mathscr{H}(f,\mathscr{I}^*)$. But, by (ii) and (iii) $\mathscr{H}(\ ,\mathscr{I}^*)$ has an inverse functor, namely $*\circ\mathscr{H}(\mathscr{I}^{-1},\ )$, which induces a map, e'', that is inverse to e'. Elementary considerations show that both e' and e'' are continuous. Hence, e'

is an isomorphism. Therefore, e is an isomorphism. Therefore, $\mathscr{H}(\ ,\mathscr{I}^*)$ is a duality functor. This proves (iv).

(v) is an immediate consequence of (i), (iii), and the obvious fact that $\mathscr{H}(\mathscr{I},\ )$ and $\mathscr{H}(\mathscr{J},\ )$ are equivalent as functors if and only if the ideals, $\mathscr{I}$ and $\mathscr{J}$, are in the same ideal class of $\mathscr{R}$. Theorem 10 is proved.

Remark. The importance of the additional functors given by Theorem 11 is considerably undermined by the following fact. If $\mathscr{R} = \mathscr{O}$ is a ring of integers of a global field and $\mathscr{I}$ is an ideal of $\mathscr{O}$, then the restriction of $\mathscr{H}(\mathscr{I},\ )$ to the category of $\mathscr{O}$-modules of the form, $M_3 \oplus \mathscr{R}$, (i.e., with trivial torsion-free discrete and compact divisible parts) is equivalent to the identity functor on that category. We leave the verification of this to the reader.

8. Local Linear Compactness

In this paper we have generalized the structure theory of locally compact abelian groups to locally compact modules over a ring of integers of a global field. One would like to generalize this structure theory to locally compact modules over a wider class of rings. However, for a ring even as simple as $\mathbf{Q}[t]$ (where $\mathbf{Q}$ is the field of rational numbers and t is an indeterminant), there is no simple generalization of Theorem 2, the fundamental structure theorem.

However, we can generalize the structure theory if we abandon the requirement of local compactness and in its stead require *local linear compactness*, a notion introduced by Lefshetz [9] and developed somewhat further by others (cf. [1, 4, 8]).

Let F' be any field, regarded as having the discrete topology. Let M be a topological vector space over F'. We call M a *linearly compact space* over F' if it is isomorphic to the product (with the product topology) of copies of F'. We call M *locally linearly compact* if it has an open linearly compact subspace.

There is a duality theory for locally linear compact spaces over F' completely analogous to the Pontryagin duality theory (but more easily proved).

Analogous to what we did for finite fields, let $\mathscr{Z}'$ be the ring of polynomials in one variable with constant field F'. Let k' be a finite separable extension of the field of fractions of $\mathscr{Z}'$, and let $\mathscr{O}'$ be the integral closure of $\mathscr{Z}'$ in k'. Call M an *$\mathscr{O}'$-module* if it is a topological module over $\mathscr{O}'$ which, as a vector space over F', is locally linearly compact.

With these definitions all of the theorems of this paper still hold true with only the most minor modifications in the proofs. In particular, the analogue of Theorem 3 gives a nontrivial decomposition theorem for any continuous operator on a locally linearly compact space. Also, the analogues of the theorems of Section 5 give characterizations of the adele rings of function fields in one variable over arbitrary constant field.

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